Translation of S4 and K5 into GF and 2VAR

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Introduction

Program: Try to understand modal logics through first order logic by viewing them as subsets.

Possible views are the guarded fragment, and the 2-variable fragment.

The guarded fragment was introduced because the 2-variable fragment was insufficient. No hope for generalizations, no tree-models, some logics obviously have more variables.

The guarded fragment seems to be the right fragment, but it cannot handle certain frame properties.

Because of this, various (really hard) generalizations have been proposed.

I give a simple translation of modal logics S4 and K4 into the guarded fragment, and the 2-variable fragment.

Modal Structures

A *modal structure* is a tuple of the form $I = (S, R, v)$, satisfying the following:

- S is a non-empty set, the *domain* or the set of *possible worlds*.
- $R \subseteq S \times S$ is a binary relation, the *accessibility relation*.
- v is a (partial) function, that assigns a truth value to some modal atoms in F in each $s \in S$. The first argument is the world, and the second the atom.

I is an $S4$-structure if relation R is transitive and reflexive.

I is an $S5$-structure if relation R is an equivalence relation.

I is a $K5$-structure if relation R is Euclidean, i.e.

$$\forall xyz \ (R(x,y) \wedge R(x,z) \rightarrow R(y,z) \ ).$$

Guarded Fragment, Two Variable Fragment

Both fragments are without function symbols, and (here) without equality.

The guarded fragment is obtained by restricting quantification to the following forms:

$$\forall \overline{x}(a \rightarrow A), \qquad \exists \overline{x}(a \land A).$$

In both, $\overline{x}$ is a sequence of variables, A is guarded, and a is an atom containing the free variables of A.

The 2-variable fragment is the subset of first order logic that can be written down using only 2 variables.

Modal logic K can be translated into these fragments as follows: Let F be a modal formula. Let α and β be two first order variables. We define the following translation function $T(F, \alpha, \beta)$.

- For an atom a, the translation $T(a, \alpha, \beta)$ is defined as $a(\alpha)$.

- For a negated atom $\neg a$, the translation $T(\neg a, \alpha, \beta)$ is defined as $\neg a(\alpha)$.

- For a formula of the form $A \wedge B$, the translation $T(A \wedge B, \alpha, \beta)$ equals $T(A, \alpha, \beta) \wedge T(B, \alpha, \beta)$.

- For a formula of the form $A \vee B$, the translation $T(A \vee B, \alpha, \beta)$ equals $T(A, \alpha, \beta) \vee T(B, \alpha, \beta)$.

The modal cases:

- For a formula of the form $\Diamond A$, the translation $T(\Diamond A, \alpha, \beta)$ equals $\exists\beta[R(\alpha, \beta) \wedge T(A, \beta, \alpha)]$.

- For a formula of the form $\Box A$, the translation $T(\Box A, \alpha, \beta)$ equals $\forall\beta[R(\alpha, \beta) \rightarrow T(A, \beta, \alpha)]$.

Additional frame properties can be added after the translation, when they are in the fragment of choice.

Many frame properties do not fit into the fragments. The most important of these is transitivity.

Various generalizations have been proposed for handling transitive frames, e.g. the 2-variable guarded fragment with transitive relations. It is shown decidable by reduction Rabin's tree theorem.

Another approach is GF with fixed points.

Guarded Fragment with Fixed Points

If one cannot force the modal structure to be transitive, then let's pretend that it is transitive:

Replace $\Diamond$ and $\Box$ by $\Diamond^*$ and $\Box^*$.

Extend GF by *fixed point operators*:

Least fixed point:

$$\mu(P(x_1, \ldots, x_n) : A(P, x_1, \ldots, x_n)).$$

Greatest fixed point:

$$\nu(P(x_1, \ldots, x_n) : A(P, x_1, \ldots, x_n)).$$

Formula A has only positive occurrences of P, and A is guarded.

Reachable from x through predicate R :

$$\mu(P(y) : R(x, y) \vee \exists z(R(z, y) \wedge P(z))).$$

μ-Calculus

How the translation works, can best be understood through μ-calculus. The μ-calculus is modal logic with fixed point operators:

$$\mu P(\Phi(P)), \quad \nu P(\Phi(P)).$$

Meaning is:
(μ) : Smallest predicate P for which $\Phi(P) \rightarrow P$.
(ν) : Largest predicate P for which $\Phi(P) \rightarrow P$.

Replace

$$\Box_{K4}(A) = \nu P(\Box(A \wedge P)),$$

$$\Diamond_{K4}(A) = \mu P(\Diamond(A \vee P)).$$

In the GF:

$$\nu\ p(x) : \forall y(r(x,y) \rightarrow (a(y) \wedge p(y))).$$

$$\mu\ p(x) : \exists y(r(x,y) \wedge (a(y) \vee p(y))).$$

Problem:

This works, but it is really too much violence for these poor, little modal logics.

Translation for S4

Definition
A modal formula is in *negation normal form* (NNF) if it does not contain the $\rightarrow$, the $\leftrightarrow$, and all negations $\neg$ occur in subformulae of the form $\neg a$, where a is an atom.

Examples:

$$\Box a \vee b, \quad \Diamond(a \vee \neg b),$$

The translation could be done without NNF, but would be a bit more complicated.

All translation steps are the same, except the modal steps:

- For a formula of the form $\Diamond A$, the translation $T(\Diamond A, \alpha, \beta)$ equals

$$\exists \beta [R(\alpha, \beta) \land T(A, \beta, \alpha)].$$

- For a formula of the form $\Box A$, the translation $T(\Box A, \alpha, \beta)$ equals

$$X(\alpha) \land \forall \alpha \beta [R(\alpha, \beta) \rightarrow (X(\alpha) \rightarrow X(\beta))] \land$$

$$\forall \alpha [X(\alpha) \rightarrow T(A, \alpha, \beta)].$$

  Here X is a unary predicate symbol that occurs nowhere else.

Theorem:
For every modal formula F, the result $T(F, \alpha, \beta)$ is in the 2 variable fragment, and in the guarded fragment.

Theorem:
Let F be a modal formula, that has an $S4$-model. The translation $T(F, \alpha, \beta)$ has a FOL-model. (If $\alpha \neq \beta$.)

Proof:
Let $I = (S, R, v)$ be the $S4$-model of F, so there is an $s \in S$, such that $I \models F[s]$. We construct the FOL-model $(S, [\ ])$, which has the same domain as I. The valuation [ ] is defined from:

- $[R] = \{(s_1, s_2) \mid R(s_1, s_2)\}$.

- For a unary predicate symbol a in $T(F, \alpha, \beta)$, that is the translation of an atom a in F, put $[a] = \{s \mid v(s, a) = \mathbf{t}\}$.

- For a unary predicate symbol X in $T(F, \alpha, \beta)$, that was introduced when translating $\Box A$, put $[X] = \{s \mid I \models \Box A[s]\}$.

Theorem: Let F be a modal formula. Assume that its translation $T(F, \alpha, \beta)$ has a FOL-model. Then F has an $S4$-model.

Proof: Let $(D, [\,]) \models T(F, \alpha, \beta)$. The $S4$-model (D, R, v) has the same domain.

- First define R' from: $R'(d_1, d_2)$ iff $(d_1, d_2) \in [R]$. Let R be the transitive, reflexive closure of R'.

- For each predicate name a, put $v(d, a) = \mathbf{t}$ iff $d \in [a]$.

Some Other Logics

The T-translation can be modified for $K4, B, T, S5$, by changing the translations for $\Box A$. We give some other translations here, the X is always a fresh name.

K Define $T(\Box, \alpha, \beta)$ as

$$X(\alpha) \wedge \forall\alpha\beta[R(\alpha,\beta) \rightarrow X(\alpha) \rightarrow T(A,\beta,\alpha)].$$

K4 Define $T(\Box A, \alpha, \beta)$ as

$$X(\alpha) \wedge \forall\alpha\beta[R(\alpha,\beta) \rightarrow X(\alpha) \rightarrow X(\beta)] \wedge \forall\alpha\beta[R(\alpha,\beta)$$

S5 Define $T(\Box A, \alpha, \beta)$ as

$$X(\alpha) \wedge \forall\alpha[X(\alpha) \rightarrow T(A,\alpha,\beta)] \wedge \forall\alpha\beta[R(\alpha,\beta) \rightarrow X(\alpha$$

$$\forall\alpha\beta[R(\beta,\alpha) \rightarrow X(\beta) \rightarrow X(\alpha)].$$

Euclidean Frames, K5

A relation R is *Euclidean* if it satisfies the following property:

$$R(x,y) \wedge R(x,z) \Rightarrow R(y,z).$$

This is clearly equivalent with the following

$$R(x,y) \wedge R(x,z) \Rightarrow R(y,z) \text{ and } R(z,y).$$

The translation steps are unchanged for all operators except $\Box$. For $\Box A$, the translation $T(\Box A, \alpha, \beta)$ becomes

$$\forall \beta [R(\alpha, \beta) \rightarrow T(A, \beta, \alpha)] \wedge$$

$$\forall \beta [R(\beta, \alpha) \rightarrow X(\beta)] \wedge$$

$$\forall \alpha \beta [R(\alpha, \beta) \wedge X(\alpha) \rightarrow X(\beta)] \wedge$$

$$\forall \alpha \beta [R(\beta, \alpha) \wedge X(\alpha) \rightarrow X(\beta)] \wedge$$

$$\forall \alpha \beta [R(\alpha, \beta) \wedge X(\alpha) \rightarrow T(A, \beta, \alpha)].$$

(As given here, the translation has exponential complexity, but this can be easily repaired)

Correctness of the translation can be proven through the following lemma. The X predicate is intended to be true for the e_i.

Lemma:
Let R be binary relation on some set D. Let $\overline{R}$ be the Euclidean closure of R. For each $d_1, d_2 \in D$, the following 2 are equivalent:

1. $\overline{R}(d_1, d_2)$,

2. Either $R(d_1, d_2)$ or there exist $e_1, \ldots, e_n \in D, (n > 1)$, with $R(e_1, d_1)$, $R(e_n, d_2)$, and for each $i, (1 \leq i < n)$, either $R(e_i, e_{i+1})$ or $R(e_{i+1}, e_i)$.

Theorem:
Let F be a modal formula. F has a Euclidean model $I = (S, R, v)$, iff the translation $T(F, \alpha, \beta)$ has a FOL-model.

Characterization of the Strength of the Method

At this moment, I do not have any formal results on the strength I give only an intuition:

Observation: The method works by extending the range of $\Box$, by assigning monadic predicate letters to inbetween-worlds.

This looks a bit like a NDFA.

Guess:
The method works for modal logics, for which the frame property is 'regular'.

Let $\Sigma = \{a, b\}$ be a two letter alphabet. Let a denote following R, let b denote going against R.

Then one can view the frame properties as grammars:

Transitivity:

$$a \Rightarrow aa, \quad b \Rightarrow bb.$$

Reflexivity:

$$a \Rightarrow \lambda, \quad b \Rightarrow \lambda.$$

Euclideannes:

$$a \Rightarrow ba, \quad b \Rightarrow ba.$$

All these languages are indeed regular. For Euclideannes, the regular expression is $a(a|b)^*b$.

A probable candidate for which the method would not work, is the (artificial) non-regular frame property:

$$\forall xyzt(R(x,y) \wedge R(x,z) \wedge R(z,t) \rightarrow R(y,t)).$$

Its grammar is

$$a \Rightarrow baa, \qquad b \Rightarrow abb.$$

That this is non-regular can be seen from the strings $b^i a a^i, (i > 0)$

Conclusions

We give a nice (and surprising) embedding of some modal logics into GF and the 2-variable fragment.

We gave a reasonable guess, for which modal logics the method works.

The idea could be easily generalized to multi-modal logics.